AQA

A-level
FURTHER MATHEMATICS
7367/1
Paper 1

Mark scheme

June 2020

Version: 1.0 Final Mark Scheme

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Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Examiner.

It must be stressed that a mark scheme is a working document, in many cases further developed and expanded on the basis of students' reactions to a particular paper. Assumptions about future mark schemes on the basis of one year's document should be avoided; whilst the guiding principles of assessment remain constant, details will change, depending on the content of a particular examination paper.

Further copies of this mark scheme are available from aqa.org.uk

Mark scheme instructions to examiners

General

The mark scheme for each question shows:

- the marks available for each part of the question
- the total marks available for the question
- marking instructions that indicate when marks should be awarded or withheld including the principle on which each mark is awarded. Information is included to help the examiner make his or her judgement and to delineate what is creditworthy from that not worthy of credit
- a typical solution. This response is one we expect to see frequently. However credit must be given on the basis of the marking instructions.

If a student uses a method which is not explicitly covered by the marking instructions the same principles of marking should be applied. Credit should be given to any valid methods. Examiners should seek advice from their senior examiner if in any doubt.

Key to mark types

M	mark is for method
R	mark is for reasoning
A	mark is dependent on M marks and is for accuracy
B	mark is independent of M marks and is for method and accuracy
E	mark is for explanation
F	follow through from previous incorrect result

Key to mark scheme abbreviations

CAO	correct answer only
CSO	correct solution only
ft	follow through from previous incorrect result
'their'	indicates that credit can be given from previous incorrect result
AWFW	anything which falls within
AWRT	anything which rounds to
ACF	any correct form
AG	answer given
SC	special case
OE	or equivalent
NMS	no method shown
PI	possibly implied
sf	significant figure(s)
dp	decimal place(s)

Examiners should consistently apply the following general marking principles:

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

Diagrams

Diagrams that have working on them should be treated like normal responses. If a diagram has been written on but the correct response is within the answer space, the work within the answer space should be marked. Working on diagrams that contradicts work within the answer space is not to be considered as choice but as working, and is not, therefore, penalised.

Work erased or crossed out

Erased or crossed out work that is still legible and has not been replaced should be marked. Erased or crossed out work that has been replaced can be ignored.

Choice

When a choice of answers and/or methods is given and the student has not clearly indicated which answer they want to be marked, mark positively, awarding marks for all of the student's best attempts. Withhold marks for final accuracy and conclusions if there are conflicting complete answers or when an incorrect solution (or part thereof) is referred to in the final answer.

AS/A-level Maths/Further Maths assessment objectives

AO		Description
AO1	AO1.1a	Select routine procedures
	AO1.1b	Correctly carry out routine procedures
	AO1.2	Accurately recall facts, terminology and definitions
AO2	AO2.1	Construct rigorous mathematical arguments (including proofs)
	AO2.2a	Make deductions
	AO2.2b	Make inferences
	AO2.3	Assess the validity of mathematical arguments
	AO2.4	Explain their reasoning
	AO2.5	Use mathematical language and notation correctly
AO3	AO3.1a	Translate problems in mathematical contexts into mathematical processes
	AO3.1b	Translate problems in non-mathematical contexts into mathematical processes
	AO3.2a	Interpret solutions to problems in their original context
	AO3.2b	Where appropriate, evaluate the accuracy and limitations of solutions to problems
	AO3.3	Translate situations in context into mathematical models
	AO3.4	Use mathematical models
	AO3.5a	Evaluate the outcomes of modelling in context
	AO3.5b	Recognise the limitations of models
	AO3.5c	Where appropriate, explain how to refine models

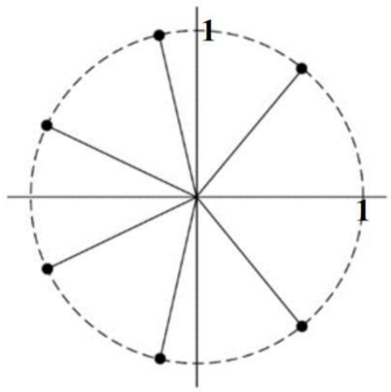
Q	Marking Instructions	AO	Marks	Typical Solution
1	Circles $\int_0^1 \sqrt{x}\,\mathrm{d}x$	2.2a	B1	$\int_0^1 \sqrt{x}\,\mathrm{d}x$
	Total		**1**	

Q	Marking Instructions	AO	Marks	Typical Solution
2	Circles $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$	1.2	B1	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$
	Total		**1**	

Q	Marking Instructions	AO	Marks	Typical Solution
3	Ticks $b < 0$ and $c < 0 \Rightarrow \alpha > 0$ and $\beta > 0$	2.2a	B1	$b < 0$ and $c < 0 \Rightarrow \alpha > 0$ and $\beta > 0$
	Total		**1**	

Q	Marking Instructions	AO	Marks	Typical Solution
4(a)	Identifies $1+3i$ as a second root of the quartic equation.	1.1b	B1	$z = 1 + 3\mathrm{i}$ is another root. $z^2 - 2z + 10$ is a factor. $(z^2 - 2z + 10)(z^2 + bz + 8)$ $\equiv z^4 - 2z^3 + pz^2 + rz + 80$ Comparing z^3-terms gives $b = 0$ ∴ the quartic is $(z^2 - 2z + 10)(z^2 + 8)$
	Uses a pair of conjugate roots to find a quadratic factor.	1.1a	M1	
	Finds one correct quadratic factor.	1.1b	A1	
	Correctly expresses the quartic as the product of two quadratic factors.	1.1b	A1	
4(b)	Substitutes $1+3i$ or $1-3i$ or one of their roots from their factorisation in (a) into the quartic equation and compares Re and Im parts or compares the coefficients of z^2 and z from the (possibly partial) expansion of their product of quadratics with the given quartic.	1.1a	M1	$z^4 - 2z^3 + 18z^2 - 16z + 80 = 0$ $p = 18$ $r = -16$
	Finds the correct values of $p = 18$ and $r = -16$	1.1b	A1	
	Total		**6**	

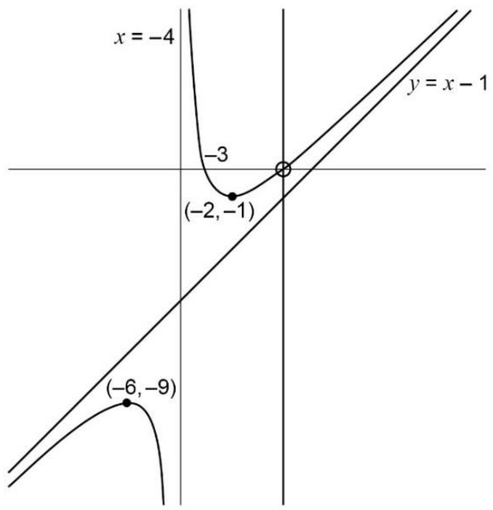
Q	Marking Instructions	AO	Marks	Typical Solution
5(a)	Forms an expression for the distance of a general point from the line $x = 2$. Condone $(x-2)$ or $(2-x)$	1.1b	B1	$\sqrt{(x-5)^2+y^2} = 2\|x-2\|$ $(x-5)^2 + y^2 = 4(x-2)^2$ $x^2 - 10x + 25 + y^2 = 4x^2 - 16x + 16$ $3x^2 - 6x - y^2 = 9$ $x^2 - 2x - \frac{y^2}{3} = 3$ $(x-1)^2 - 1 - \frac{y^2}{3} = 3$ $(x-1)^2 - \frac{y^2}{3} = 4$
	Forms an expression for the (squared) distance of a general point from (5, 0)	1.1b	B1	
	Forms an equation in x and y for the locus, of the form: distance from (5, 0) = 2 x distance from $x = 2$	3.1a	M1	
	Simplifies their equation to a quadratic in x and y with coefficient of x equal to 1	1.1a	M1	
	Completes a rigorous argument to show $(x-1)^2 - \frac{y^2}{3} = 4$ AG	2.1	R1	
5(b)	States that there is a horizontal (PI by vector with no y-component) translation.	2.2a	B1	Translation by $\begin{bmatrix}1\\0\end{bmatrix}$ Stretch parallel to y-axis. Scale factor $\sqrt{3}$
	Describes translation as $\begin{bmatrix}1\\0\end{bmatrix}$ OE	2.5	B1	
	States that there is a stretch parallel to the y –axis.	2.2a	B1	
	States that the scale factor is $\sqrt{3}$ FT their q (or $\sqrt{q}$ if no value of q given.)	2.5	B1F	
	Total		**9**	

Q	Marking Instructions	AO	Marks	Typical Solution
6(a)	Completes a rigorous argument to show that w^n satisfies the equation $z^7 = 1$	2.1	R1	$(w^n)^7 = w^{7n} = (w^7)^n = 1^n = 1$ $\therefore w^n$ satisfies the equation $z^7 = 1$
6(b)	Deduces that the LHS is the sum of the roots of $z^7 = 1$or factorises $w^7 - 1$ or uses the sum of a geometric series with values for n and a.	2.2a	M1	The roots of $z^7 - 1 = 0$ are 1, w, w^2, w^3, w^4, w^5, and w^6 z^6 term $= 0 \therefore$ sum of roots $= 0$ and $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$ as required.
	Completes a rigorous argument to show $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$	2.1	R1	
6(c)	Shows the six required points or vectors with correct arguments, approximately correctly spaced and approximately symmetric in the real axis.	1.1a	M1	
	Clearly shows that points/vectors have modulus 1. PI by "1" marked on an axis. Labelling of points not required.	1.1b	A1	
6(d)	States that $w = \cos\frac{2\pi}{7} + \cdots$	1.1b	B1	$w = \cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}$ and $w^6 = \cos(-\frac{2\pi}{7}) + \mathrm{i}\sin(-\frac{2\pi}{7})$ $= \cos\frac{2\pi}{7} - \mathrm{i}\sin\frac{2\pi}{7}$ which is the conjugate of w. $\therefore w + w^6 = 2\cos\frac{2\pi}{7}$ Similarly $w^2 + w^5 = 2\cos\frac{4\pi}{7}$ and $w^3 + w^4 = 2\cos\frac{6\pi}{7}$ From part (b), $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$ $(w + w^6) + (w^2 + w^5) + (w^3 + w^4) = -1$ $2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = -1$ $\therefore \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$ as required.
	Explains that complex conjugate pairs have the same real part.	2.4	E1	
	Deduces that a sum of pairs of powers of w equals twice the cosine of a correct angle.	2.2a	M1	
	Completes a rigorous argument, using $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$ to show $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$	2.1	R1	
	Total		**9**	

Q	Marking Instructions	AO	Marks	Typical solution
7(a)	Shows correctly that $k = 4.5$ gives a determinant of 0 or shows $k = 4.5$ from solving det $M = 0$	1.1b	B1	$0 = (4k+1)[4(3-k)+6] + 3[4(k-1)-14]$ $+ (k-5)[-3(k-1)-7(3-k)]$ $0 = -12k^2 + 42k + 54$ $k = 4.5,\ k = -1$
	Correctly expands the determinant of the matrix and equates to 0 Condone misread of $-3y$ as $+3y$	1.1a	M1	
	Obtains $k = -1$	1.1b	A1	
7(b)	When $k = 4.5$, clearly shows or explains that the system of equations is consistent, using equations of planes or augmented matrix form. Must state the system is consistent.	3.1a	B1	When $k = 4.5$ matrix becomes: $\begin{array}{ccc\|c} 19 & -3 & -0.5 & 3 \\ 3.5 & -1.5 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}$ The system of equations is consistent. Two planes are the same and intersect the third plane in a line. When $k = -1$ matrix becomes: $\begin{array}{ccc\|c} 0 & 0 & 0 & 66 \\ -2 & 4 & 2 & 1 \\ 11 & -11 & 0 & 0 \end{array}$ The system of equations is inconsistent. The three planes form a prism.
	States that two planes are the same and intersect the third plane.	3.2a	B1	
	When $k = -1$, completes appropriate working to find the consistency of the system using their k.	3.1a	M1	
	States that the system is inconsistent and that the three planes form a prism. CSO	3.2a	A1	
	Total		**7**	

Q	Marking Instructions	AO	Marks	Typical Solution
8	Defines three roots in arithmetic progression. PI by a correct use of arithmetic series sum in terms of α and β	3.1a	M1	
	Uses one of Vieta's laws: $\alpha+\beta+\gamma=3$ $\alpha\beta+\beta\gamma+\gamma\alpha=\frac{-13}{4}$ $\alpha\beta\gamma=\frac{-k}{4}$	3.1a	M1	Let the roots be $\alpha-D$, α and $\alpha+D$ $\Sigma\alpha$: $\alpha-D+\alpha+\alpha+D=-\left(\frac{-12}{4}\right)$ $3\alpha=3$ $\alpha=1$ $\Sigma\alpha\beta$: $(1-D)(1)+(1)(1+D)+(1+D)(1-D)=\frac{-13}{4}$ $1-D+1+D+1-D^2=\frac{-13}{4}$ $3+\frac{13}{4}=D^2\Rightarrow D=\pm\frac{5}{2}$ $x=-1.5, 1, 3.5$
	Obtains 1 as a root of the equation (this might be $\alpha+D=1$ without clear realisation that 1 is therefore a root)	1.1b	A1	
	Substitutes their root of 1 in the cubic equation to find k or uses another of Vieta's laws with their root of 1 substituted.	3.1a	M1	
	Obtains the value of k or solves their equation to find D or the other roots. Allow one slip.	1.1b	A1F	
	Correctly obtains all three roots: -1.5, 1, 3.5	1.1b	A1	
	Total		**6**	

Q	Marking Instructions	AO	Marks	Typical Solution
9(a)	Forms an equation $\frac{x(x+3)}{x+4} = {'k'}$ or Differentiates using quotient rule.	3.1a	M1	
	Rearranges their equation into a quadratic in x. or Obtains the correct $f'(x) = \frac{(x+4)\times(2x+3)-(x^2+3x)\times 1}{(x+4)^2}$.	1.1a	M1	
	Explains that the discriminant of this quadratic < 0 or Explains that because there is a vertical asymptote the minimum is higher up the graph than the maximum and the two turning points lie on different branches of the graph.	2.4	E1	Assume $y = k$ meets $y = \frac{x(x+3)}{x+4}$ Then $k(x+4) = x(x+3)$ $kx + 4k = x^2 + 3x$ $x^2 + (3-k)x - 4k = 0$ $\Delta < 0 \therefore (3-k)^2 - 4(1)(-4k) < 0$ $k^2 + 10k + 9 < 0$ $(k+9)(k+1) < 0$ $-9 < k < -1$ So $f(x)$ does not take any values in the interval $(-9, -1)$
	Forms a quadratic equation or inequality in 'k' from their discriminant. or Equates their $f'(x)$ or its numerator to 0 and solves	1.1a	M1	
	Completes a rigorous argument to show that $f(x)$ does not take any values in the interval $(-9, -1)$. Condone $-9 < k < -1$	2.1	R1	

Q	Marking Instructions	AO	Marks	Typical Solution
9(b)	Substitutes their -9 or -1 into $y = f(x)$ and forms a quadratic in x or Differentiates using the quotient rule and equates their $f'(x)$ or its numerator to 0	1.1a	M1	Stationary points $(-6, -9)$ and $(-2, -1)$
	Finds the coordinates of both stationary points	1.1b	A1F	
9(c)	Divides the numerator by $x + 4$ and obtains $f(x) = x + \cdots$	3.1a	M1	$f(x) = \dfrac{x^2 + 3x}{x + 4} = \dfrac{(x + 4)(x - 1) + 4}{x + 4}$ $= x - 1 + \dfrac{4}{x + 4}$ Asymptote is $y = x - 1$
	Obtains the correct equation of the asymptote $y = x - 1$	1.1b	A1	
9(d)	Draws a curve asymptotic to $x = -4$ or their $y = x - 1$	1.1b	B1F	
	Draws a curve with two branches asymptotic to their asymptotes	1.1b	B1F	
	Draws one branch of the curve in the correct position	1.1b	M1	
	Deduces the correct curve and draws a fully correct sketch, with roots at -3 and 0 FT their oblique asymptote	2.2a	A1F	
	Total		**13**	

Q	Marking Instructions	AO	Marks	Typical Solution
10(a)	Selects a method to solve the differential equation by finding an integrating factor.	3.1a	M1	$P = \frac{2}{x} \Rightarrow \int P\mathrm{d}x = 2\ln x$ Integrating factor $= \mathrm{e}^{\int P\mathrm{d}x} = x^2$
	Obtains the correct integrating factor $= x^2$	1.1b	A1	
	Multiplies the differential equation by their integrating factor	1.1a	M1	$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = \frac{x(x+3)}{(x-1)(x^2+3)}$ $\frac{\mathrm{d}}{\mathrm{d}x}(x^2y) = \frac{x^2+3x}{(x-1)(x^2+3)}$
	Correctly integrates their LHS to obtain their x^2y	1.1b	A1F	
	Splits their RHS into appropriate partial fractions, including numerators in the correct form.	3.1a	M1	$\frac{x^2+3x}{(x-1)(x^2+3)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+3}$ $\equiv \frac{x^2+3+3x-3}{(x-1)(x^2+3)} \equiv \frac{1}{x-1} + \frac{3}{x^2+3}$
	Obtains partial fractions of the form $\frac{A}{x-1} + \frac{Bx+C}{x^2+3}$ with values for A, B, C, with no extra fractions	1.1a	M1	
	Integrates $\frac{1}{x^2+3}$ to obtain $k\tan^{-1}\frac{x}{\sqrt{3}}$	1.1b	B1	$x^2y = \int\left(\frac{1}{x-1} + \frac{3}{x^2+3}\right)\mathrm{d}x$ $x^2y = \ln(x-1) + \frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + c$ or $y = \frac{1}{x^2}\left(\ln(x-1) + \sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}} + c\right)$
	Obtains a completely correct expression for the general solution: $xy = \ln(x-1) + \sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}} + c$ Condone omission of c. ACF	1.1b	A1	
10(b)	Substitutes $(3,0)$ into their general solution, with a constant of integration, and solves to find a value for it.	1.1a	M1	$0 = \frac{1}{9}(\ln 2 + \sqrt{3}\tan^{-1}\sqrt{3} + c)$ $c = -\ln 2 - \frac{\pi\sqrt{3}}{3}$ $y = \frac{1}{x^2}\left(\ln(x-1) + \sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}} - \ln 2 - \frac{\pi\sqrt{3}}{3}\right)$
	Obtains the correct solution in the form $y = \mathrm{f}(x)$. FT their general solution from (a). Condone their correct c as a decimal.	1.1b	A1F	
	Total		**10**	

Q	Marking Instructions	AO	Marks	Typical Solution
11(a)(i)	Explains why l_1 and l_2 are parallel, with reference to the direction vector of l_1 being $-1\times$ the direction vector of l_2.	2.4	E1	$\begin{bmatrix}-2\\1\\-3\end{bmatrix}=-\begin{bmatrix}2\\-1\\3\end{bmatrix}$ The direction vectors for l_1 and l_2 are multiples of each other, so l_1 and l_2 are parallel lines.
11(a)(ii)	Forms an expression for a vector from a point on l_1 to a point on l_2. PI by $\sqrt{89}$	3.1a	M1	Given a point A on l_1 and a point B on l_2, distance $=\lvert AB\sin\theta\rvert$, where either line makes an angle θ with AB.
	Obtains a correct $\overrightarrow{AB}$. PI by $\sqrt{89}$ or Obtains a correct parametrised form for $\overrightarrow{PQ}$	1.1b	A1	$A(1,5,-1), B(-3,2,7);\ \overrightarrow{AB}=\begin{bmatrix}-4\\-3\\8\end{bmatrix}$ Unit direction vector $\hat{l}=\frac{1}{\sqrt{14}}\begin{bmatrix}2\\-1\\3\end{bmatrix}$ $\overrightarrow{AB}\times\hat{l}=\frac{1}{\sqrt{14}}\begin{vmatrix}\mathbf{i}&-4&2\\\mathbf{j}&-3&-1\\\mathbf{k}&8&3\end{vmatrix}=\frac{1}{\sqrt{14}}\begin{vmatrix}-1\\28\\10\end{vmatrix}$
	Forms a vector product of $\overrightarrow{AB}$ and the direction vector of l_1 or l_2 or Forms a scalar product of $\overrightarrow{PQ}$ and the direction vector of l_1 or l_2	3.1a	M1	Required distance $=\frac{1}{\sqrt{14}}(\sqrt{1^2+28^2+10^2})$ = 7.95 (3 sig fig) as required. Alternative typical solution: $\overrightarrow{PQ}=\begin{bmatrix}4-2\lambda\\3+\lambda\\-8-3\lambda\end{bmatrix}$ $0=\begin{bmatrix}4-2\lambda\\3+\lambda\\-8-3\lambda\end{bmatrix}\cdot\begin{bmatrix}-2\\1\\-3\end{bmatrix}$
	Uses either distance $=k\left\lvert\overrightarrow{AB}\times\begin{bmatrix}2\\-1\\3\end{bmatrix}\right\rvert$ or $\overrightarrow{PQ}.\begin{bmatrix}2\\-1\\3\end{bmatrix}=0$	1.1a	M1	$0=-8+4\lambda+3+\lambda+24+9\lambda$ $\lambda=\frac{-19}{14}$ $\overrightarrow{PQ}_{min}=\begin{bmatrix}47/7\\23/14\\-55/14\end{bmatrix}$
	Completes a rigorous argument to show that distance = 7.95 **AG**	2.1	R1	Required distance $=\sqrt{(47/7)^2+(23/14)^2+(55/14)^2}$ = 7.95 (3 sig fig)

Q	Marking Instructions	AO	Marks	Typical Solution
11(b)	Expresses the two lines in parametric form. Condone the same parameter for both lines.	3.1a	M1	$\mathbf{r} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} + \mu \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ $\mathbf{r} = \begin{bmatrix} -5 \\ 12 \\ -4 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 0 \\ 9 \end{bmatrix}$ $x \quad 1 - 2\mu = -5 + 4\lambda$ $y \quad 5 + \mu = 12$ $z \quad -1 - 3\mu = -4 + 9\lambda$ Eqn. 2 $\Rightarrow \mu = 7$ Sub. in eqn. 1 giving $\lambda = -2$ $\mu = 7$ and $\lambda = -2$ satisfy eqn. 3, and the lines meet at (-13,12, -22)
	Equates the two lines with two different parameters.	1.1a	M1	
	Solves two equations, from components of the vectors, simultaneously.	1.1b	M1	
	Verifies that the third components are equal.	2.2a	A1	
	Completes a rigorous argument by concluding that the third equation is satisfied and finding the coordinates of the point of intersection.	2.1	R1	
	Total		**11**	

Q	Marking Instructions	AO	Marks	Typical Solution
12(a)	Expresses x in terms of y	3.1a	M1	Let $y = \cosh^{-1}\left(\frac{x}{a}\right)$; then $x = a\cosh y$ $x = \frac{a}{2}(e^y + e^{-y})$ $\frac{2x}{a} = e^y + e^{-y}$ $\times\, e^y$: $e^{2y} - \frac{2x}{a}e^y + 1 = 0$ $e^y = \frac{\frac{2x}{a} \pm \sqrt{\frac{4x^2}{a^2} - 4}}{2}$ $e^y = \frac{x \pm \sqrt{x^2 - a^2}}{a}$ Product of roots = 1, so one root is greater than 1 and the other is less than 1. $y \geq 0$ by definition of $\cosh^{-1}$ $\therefore e^y \geq 1$ So we choose the larger root. $e^y = \frac{x + \sqrt{x^2 - a^2}}{a}$ and $\cosh^{-1}\left(\frac{x}{a}\right) = \ln\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right)$ as required.
	Recalls the exponential form of cosh.	1.1b	B1	
	Forms a quadratic equation in e^y	1.1a	M1	
	Solves their quadratic equation in e^y, to obtain two solutions.	1.1b	A1F	
	Explains that inverse cosh is defined to be non-negative or explains that $\frac{x - \sqrt{x^2 - a^2}}{a}$ is <1 or shows that $\frac{x - \sqrt{x^2 - a^2}}{a} = \frac{1}{\frac{x + \sqrt{x^2 - a^2}}{a}}$	2.4	M1	
	Completes a rigorous argument to show the required result, with reference to the larger root of the quadratic being the valid one with a clear reason.	2.1	R1	
12(b)	States $\ln\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) = \ln\left(x + \sqrt{x^2 - a^2}\right) - \ln(a)$	2.4	E1	$\ln\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right)$ $= \ln\left(x + \sqrt{x^2 - a^2}\right) - \ln(a)$ $\therefore c = -\ln(a)$
	States $c = -\ln(a)$.	2.3	E1	
	Total		**8**	

Q	Marking Instructions	AO	Marks	Typical solution
13(a)	Deduces the total extension of both strings is 3. PI	2.2a	B1	
	Finds an expression for the tension in one string in terms of c and an extension.	3.4	B1	
	Forms a two-term force equation at equilibrium.	3.4	M1	
	Obtains the two correct equilibrium extensions.	1.1b	A1	
	Forms an equation of motion in terms of a general displacement with at least one correct extension FT their equilibrium extensions.	3.4	M1	$3 = y_{AB} + y_{BC}$ $5cy_{AB} = cy_{BC}$ $6\,y_{AB} = 3$ $y_{AB} = \frac{1}{2}$, $y_{BC} = \frac{5}{2}$ $5c\left(\frac{1}{2} - x\right) - c\left(\frac{5}{2} + x\right) = 3c\ddot{x}$ $\ddot{x} = -2x$ Of form $\ddot{x} = -\omega^2 x$, therefore SHM
	Obtains correct equation of motion FT their equilibrium extensions.	1.1b	A1F	
	Simplifies their equation of motion correctly to the form $\ddot{x} = -kx$ (May use a, $\frac{dv}{dt}$ or any other correct symbol for acceleration)	1.1a	M1	
	Correctly concludes that the particle moves with SHM with a clear reason from their equation of the correct form e.g. comparison with the standard form $\ddot{x} = -\omega^2 x$	2.1	R1F	

Q	Marking Instructions	AO	Marks	Typical solution
13(b)	Obtains the correct value for ω FT their final equation in (a)	1.1b	B1F	$\omega = \sqrt{2}$ $A = \frac{1}{3}$ $v^2 = 2\left(\frac{1}{3^2} - \frac{1}{4^2}\right)$ $v = 0.31$ ms^{-1}
	States or uses the correct value for the amplitude.	3.1b	B1	
	Uses a correct complete method to find the speed.	3.1b	M1	
	Obtains the correct speed with correct units FT their ω and accurate to 2 or more sf	3.2a	A1F	
	Total		**12**	

Q	Marking Instructions	AO	Marks	Typical Solution
14(a)	Writes the correct expressions for $\sinh(m+1)x$ and $\sinh(m-1)x$	1.1b	B1	$\sinh(m+1)x = \sinh mx \cosh x + \cosh mx \sinh x$ $\sinh(m-1)x = \sinh mx \cosh x - \cosh mx \sinh x$
14(b)	Subtracts $\sinh(m+1)x$ and $\sinh(m-1)x$ expressions.	3.1a	M1	$\sinh(m+1)x - \sinh(m-1)x$ $= 2\cosh mx \sinh x$ $\cancel{\sinh 2x} - \sinh 0 = 2\cosh x \sinh x$ $\cancel{\sinh 3x} - \sinh x = 2\cosh 2x \sinh x$ $\cancel{\sinh 4x} - \cancel{\sinh 2x} = 2\cosh 3x \sinh x$ … … … $\cancel{\sinh(n-1)x} - \cancel{\sinh(n-3)x}$ $= 2\cosh(n-2)x \sinh x$ $\sinh nx - \cancel{\sinh(n-2)x} = 2\cosh(n-1)x \sinh x$ $\sinh(n+1)x - \cancel{\sinh(n-1)x} = 2\cosh nx \sinh x$ $\sinh(n+1)x + \sinh nx - \sinh x =$ $2\sinh x(\cosh x + \cosh 2x + \cdots + \cosh nx)$ $= 2\sinh x C_n$ $\therefore C_n = \dfrac{\sinh(n+1)x + \sinh nx - \sinh x}{2\sinh x}$
	Uses the method of differences, with at least the first two terms shown.	3.1a	M1	
	Obtains correct terms for method of differences, with at least the first three terms and the last two.	1.1b	A1	
	Deduces that $\sum 2\cosh mx \sinh x$ is a multiple of C_n	2.2a	M1	
	Completes a rigorous argument to show the required result.	2.1	R1	
	Total		**6**	

	Paper total		**100**	